

# Approximation of Lipschitz functions by Lipschitz, $C^p$ smooth functions on weakly compactly generated Banach spaces

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**ABSTRACT.** This note corrects a gap and improves results in an earlier paper by the first named author [F3].

More precisely, it is shown that on weakly compactly generated Banach spaces  $X$  which admit a  $C^p$  smooth norm, one can uniformly approximate uniformly continuous functions  $f : X \rightarrow \mathbb{R}$  by Lipschitz,  $C^p$  smooth functions. Moreover, there is a constant  $C > 1$  so that any  $\eta$ -Lipschitz function  $f : X \rightarrow \mathbb{R}$  can be uniformly approximated by  $C\eta$ -Lipschitz,  $C^p$  smooth functions.

This provides a ‘Lipschitz version’ of the classical approximation results of Godefroy, Troyanski, Whitfield and Zizler.

## 1. Introduction

The purpose of this note is to correct a gap in the proof of the main result of [F3]. Specifically, in the original proof the function  $\phi(x) = (Sx, Tx)$  was shown to map from  $X$  into the open set  $U \subset l_\infty(\mathcal{F} \times \mathbb{N}^2) \oplus c_0(\mathcal{F} \times \mathbb{N}^2)$  as required by Haydon’s theorem (see Theorem 1 below), however; for  $F : X \rightarrow \mathbb{R}$  continuous and bounded, and  $\{x_n^K\} \subset X$  dense, it may be that the map  $x \rightarrow (F(x_n^K)(Sx)_{(K,n,m)}, F(x_n^K)(Tx)_{(K,n,m)})$  does not, as was thought in [F3]. The present paper mends this difficulty under the formally stronger hypothesis that  $X$  admit a  $C^p$  smooth norm rather than merely a Lipschitz,  $C^p$  smooth bump function. For weakly compactly generated (WCG) spaces it is unknown if these two conditions are equivalent. We note, however, that there are  $C_0(T)$  spaces, where  $T$  is a tree, which admit  $C^\infty$  smooth bump functions but no Gâteaux smooth norm (see [H2]). Despite the formally stronger hypothesis, we have otherwise improved the results from [F3]. In particular, when the function  $f$  to be approximated has convex domain, we are able to remove the condition imposed in [F3] that  $f$  be bounded, and in addition obtain stronger results when  $f$  is Lipschitz (see Theorem 4 below).

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1991 *Mathematics Subject Classification.* 46B20.

*Key words and phrases.* Smooth approximation, WCG Banach space.

We wish to point out that Theorem 4 of this paper has been used in [HJ1] to construct  $C^1$  fine approximations on WCG spaces, and in [S] to construct smooth extensions of functions from closed subspaces of WCG spaces. Other reasons for the interest in smooth, Lipschitz approximations includes their importance in the construction of deleting diffeomorphisms on Banach spaces (see e.g., [AM]), and their use in smooth variational principles on infinite dimensional Hilbert manifolds ([AFL], [AFLR]).

Let us give some background. We are considering the problem of uniformly approximating continuous, real-valued functions on Banach spaces  $X$  by certain smooth functions. This problem has a long history, beginning with the work of Kurzweil [K] and continuing through to the present (see e.g., [FM], [DGZ] and further references below). The preferred method for approaching such smooth approximation problems has been via smooth partitions of unity. Indeed, the ability to uniformly approximate arbitrary continuous functions on  $X$  by  $C^p$  smooth functions is equivalent to the existence of  $C^p$  smooth partitions of unity on  $X$  (see e.g., [DGZ]). In this vein, for Banach spaces  $X$  admitting a  $C^p$  smooth bump function (a  $C^p$  smooth function with bounded, non-empty support), the existence of  $C^p$  smooth partitions of unity has been established in fairly wide classes of spaces. For example, when  $X$  is separable this was shown by Bonic and Frampton [BF], and this was later generalized to weakly compactly generated spaces [GTWZ] (see also, [DGZ], [SS]). Recently in [HH], it was shown that in  $C(K)$  spaces, for  $K$  compact, the existence of  $C^p$  smooth bump functions and  $C^p$  smooth partitions of unity are equivalent.

One of the drawbacks of employing partitions of unity is that it is very difficult to arrange for the approximating function to possess nice properties in addition to basic smoothness. For example, if one wishes the smooth approximate to be convex or a norm, then other techniques are generally required (see e.g., [MPVZ]). The situation in which one requires the approximate to be Lipschitz as well as smooth was addressed in a series of recent papers, [F1], [F2], [AFM], [AFLR], [HJ1], [HJ2]. In particular, it was shown in [AFM] that for separable Banach spaces  $X$  admitting a Lipschitz,  $C^p$  smooth bump function, the uniform approximation of uniformly continuous, bounded, real-valued functions  $f$  by Lipschitz,  $C^p$  smooth functions  $g$  is possible (we note that the uniform continuity of  $f$  and the existence of a Lipschitz,  $C^p$  smooth bump function on  $X$  are necessary). In fact, using a simple argument motivated by [HJ1] (see also Theorem 4 below), the result from [AFM] can be used to prove: For  $X$  a separable Banach space admitting a Lipschitz,  $C^p$  bump, there exists  $C > 1$  so that for  $Y \subset X$  any subset,  $\varepsilon > 0$ , and  $\eta$ -Lipschitz function  $f : Y \rightarrow \mathbb{R}$ , there exists a  $C\eta$ -Lipschitz,  $C^p$  smooth function  $g : X \rightarrow \mathbb{R}$  with  $|f - g| < \varepsilon$  on  $Y$ . When  $Y = X$ , this result has been extended to more general range spaces in [HJ1]. In [F2] this result is shown to hold for maps having any Banach as range if we suppose that  $X$  has an unconditional Schauder basis. A related result was shown in [AFLR] where it was proven in particular that for a separable Hilbert space,

the approximate  $g$  can be chosen Lipschitz and  $C^\infty$  smooth with Lipschitz constant arbitrarily close to the Lipschitz constant of  $f$ .

Aside from the  $p = 1$  case when  $X$  is a general Hilbert or superreflexive space (see e.g., [LL], [C]), and recent results for  $X = c_0(\Gamma)$  from [HJ2], all the results stated above concerning Lipschitz,  $C^p$  smooth approximation are for separable Banach spaces. As indicated above, the purpose of this note is to extend some of these results to the nonseparable, weakly compactly generated case. In this light, for the real-valued case, the particular result of [AFM] noted previously can be seen as a ‘Lipschitz version’ of the classical approximation work of [BF], while the present paper can be seen as a ‘Lipschitz version’ of the (implicit) approximation result of [GTWZ]. In this note we also present a result on the approximation of Lipschitz functions similar in vein to the Lipschitz result derived from [AFM] described above. To our knowledge, even for the  $C^2$  smooth case in non-separable Hilbert space the results herein are new. Finally, we show how the result of [GTWZ] can be obtained from our main result. The entire proof is presented here for the sake of clarity and completeness.

## 2. Main Results

The notation we use is standard, with  $X$  typically denoting a Banach space. Smoothness here is meant in the Fréchet sense and function shall mean real-valued function.  $X$  is said to be *weakly compactly generated (WCG)* if there exists a weakly compact set  $K \subset X$  with  $\overline{\text{span}}(K) = X$ . This class includes the separable and reflexive Banach spaces.

A Banach space  $X$  is said to admit a *separable projectional resolution of the identity (SPRI)*, if for the first ordinal  $\mu$  with  $\text{card}(\mu) = \text{dens}(X)$ , there exist continuous linear projections,  $\{Q_\alpha : \alpha \in \Gamma\}$ , where  $\Gamma = [\omega_0, \mu]$ , so that if we set  $R_\alpha = (Q_{\alpha+1} - Q_\alpha) / (\|Q_{\alpha+1}\| + \|Q_\alpha\|)$ , we have,

- (i).  $Q_\alpha Q_\beta = Q_{\min(\alpha, \beta)}$
- (ii).  $(Q_{\alpha+1} - Q_\alpha)(X)$  is separable for all  $\alpha \in \Gamma$
- (iii). For all  $x \in X$ ,  $\{\|R_\alpha(x)\|\}_\alpha \in c_0(\Gamma)$
- (iv). For all  $x \in X$ ,  $x \in \overline{\text{span}\{R_\alpha(x) : \alpha < \mu\}}$

One of the keys to our result is the following fundamental theorem of Haydon,

**Theorem 1** (H1). *For any set  $L$  there exists an equivalent norm  $\|(\cdot, \cdot)\|$  on  $l_\infty(L) \oplus c_0(L)$  such that if  $U(L)$  is the open subset*

$$\left\{ (f, x) \in l_\infty(L) \oplus c_0(L) : \max\{\|f\|_\infty, \|x\|_\infty\} < \left\| |f| + \frac{1}{2}|x| \right\|_\infty \right\},$$

*then  $\|(\cdot, \cdot)\|$  is  $C^\infty$  smooth on  $U(L)$  and depends locally on only finitely many non-zero coordinates there.*

We shall also require the following deep result of Amir and Lindenstrauss (see also [T]),

**Theorem 2 (AL).** *If  $X$  is a WCG Banach space, then  $X$  admits a separable projectional resolution of the identity.*

We note that our main result is stated for WCG spaces, but it applies more generally to any Banach space admitting a separable projectional resolution of the identity. Such spaces include weakly Lindelöf determined spaces, duals of Asplund spaces, and  $C(K)$  spaces for  $K$  a Valdivia compact (see [DGZ]). We first establish our result for bounded functions, then relax this condition later for convex domains.

**Theorem 3.** *Let  $X$  be a WCG Banach space which admits a  $C^p$  smooth norm. Let  $\varepsilon > 0$ ,  $G \subset X$  an open subset and  $f : G \rightarrow \mathbb{R}$  a uniformly continuous and bounded function. Then there exists a Lipschitz,  $C^p$  smooth function  $K$  on  $G$  with  $|f(x) - K(x)| < \varepsilon$  for  $x \in G$ .*

**Proof.** To simplify the proof, we shall take  $G = X$ , leaving the slight technical adjustments to accommodate a general open subset  $G \subset X$  to the reader. In our use of Theorem 1, we may assume that for some  $A \geq 2$  we have  $\|(\cdot, \cdot)\|_\infty \leq \|(\cdot, \cdot)\| \leq A \|(\cdot, \cdot)\|_\infty$ , where  $\|(\phi, x)\|_\infty = \max\{\|\phi\|_\infty, \|x\|_\infty\}$ . As well, for simplicity we shall assume that  $f$  is 1-Lipschitz; the case where  $f$  is uniformly continuous is similar.

Fix a  $C^p$  smooth norm  $\|\cdot\|$  on  $X$ , and let  $\varepsilon \in (0, 1)$ . Because  $f$  is bounded, we may assume that  $3/4 \geq f > 1/2$  by adding a suitable positive constant and scaling. Also because  $f$  is bounded, it can be uniformly approximated within  $\varepsilon/3A$  by a simple function  $\varphi$ , and hence for the purposes of approximation it is enough to work with  $\varphi$ , which we do for the remainder of the proof. We note that by choice of  $\varepsilon$  and  $A$ , we have  $0 < \varphi \leq 1$ . Let the cardinality of the range of  $\varphi$  be  $N$ .

It will be helpful in the sequel to recall the construction of  $\varphi$  here. We evenly partition  $[1/2, 3/4]$  into subintervals  $I_i = (a_i, b_i]$  with midpoint  $m_i$  and width  $\Delta = \varepsilon_1 < \varepsilon/3A$ . Define  $\varphi(x) = \sum_{i=1}^N m_i \chi_{f^{-1}(I_i)}$ .

We may suppose that  $X$  is nonseparable (see the Remark at the end of this note), and given that  $X$  is WCG, by Theorem 2 we let  $\{Q_\alpha\}_{\alpha \in \Gamma}$  be an SPRI on  $X$ , and let  $\mathcal{F}$  be the collection of all finite, non-empty subsets of  $\Gamma$ . Our goal shall be to construct appropriate maps  $S : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2)$  and  $T : X \rightarrow c_0(\mathcal{F} \times \mathbb{N}^2)$  for use in applying Theorem 1.

Using property (ii) of an SPRI, for each  $K \in \mathcal{F}$  pick a dense sequence,  $\{x_n^K\}_{n=1}^\infty \subset X_K = \text{span}\{R_\alpha(X) : \alpha \in K\}$ . From property (iv) of an SPRI, we have that

$$\mathcal{D} = \{x_n^K : K \in \mathcal{F}, n \in \mathbb{N}\}$$

is dense in  $X$ .

In the following lemma,  $\varepsilon_1$ ,  $\varphi$ , and the  $m_i$  are defined as given above. For  $\delta > 0$ , let  $\zeta_\delta \in C^\infty(\mathbb{R}, [0, 1])$  be decreasing and Lipschitz such that  $\zeta_\delta(t) = 1$  iff  $t \leq \delta/32$  and  $\zeta_\delta(t) = 0$  iff  $t \geq \delta/16$ . Using this notation, we shall require the following technical lemma.

**Lemma 1.** *For every  $x_n^K \in \mathcal{D}$  there exists an associated  $x_{n'}^{K'} \in \mathcal{D}$  and a Lipschitz function  $\zeta_n^K \in C^\infty(\mathbb{R}, [0, 1])$  with Lipschitz constant independent of  $(K, n)$  and  $\zeta_n^K(t) = 0$  for  $t \geq \varepsilon_1$ , such that for any  $y$  with  $\|y - x_n^K\| < \delta/32 = \varepsilon_1/32^2$  (i.e.  $\delta = \varepsilon_1/32$ ) we have*

$$\begin{aligned} & \varphi(x_{n'}^{K'}) \zeta_\delta(\|y - x_n^K\|) \zeta_n^K(\|y - x_{n'}^{K'}\|) \\ &= \sup \left\{ \varphi(x_{m'}^{L'}) \zeta_\delta(\|y - x_m^L\|) \zeta_m^L(\|y - x_{m'}^{L'}\|) : (L, m) \in (\mathcal{F}, \mathbb{N}) \right\}, \end{aligned}$$

where given  $(L, m)$  we denote by  $(L', m')$  the associated pair.

**Proof.** Fix  $x_n^K \in \mathcal{D}$  and  $y \in X$  with  $\|y - x_n^K\| < \delta/32$ . We assume that  $\varphi$  is continuous at  $x_n^K$ . We leave to the reader the simple verifications required when  $\varphi$  is not continuous there. For  $i = 1, \dots, N$ , define  $\rho_i = \rho_i(K, n) = \inf\{\|x_n^K - w\| : w \in \mathcal{D}, \varphi(w) \geq m_i\}$  where  $\rho_i = \infty$  if there are no such  $w$ 's. It is clear that at least one  $\rho_i$  is finite. We also remark that  $\rho_1 = 0$ , as  $\varphi(w) = m_i \geq m_1$  for some  $i$ . Unless otherwise stated, both  $i$  and  $\rho_i$  are understood to be taken with respect to  $(K, n)$  to ease notation. We shall need the following.

**Fact.** Unless  $\rho_i = \infty$  or  $\rho_i = 0$ ,  $\rho_{i+1} \geq \rho_i + \varepsilon_1$ .

**Proof of Fact.** Set  $x = x_n^K$  and suppose that for some  $i$ , with  $\rho_i$  finite and non-zero,  $\rho_{i+1} < \rho_i + \varepsilon_1$ . Then there is a  $z_1 \in D$  with  $\varphi(z_1) = m_{i+1}$  such that  $\|z_1 - x\| = \rho_i + \varepsilon_1 - \eta$  for some  $\eta$  satisfying  $0 < \eta < \rho_i$ . Let  $z_2$  be on the line segment  $[x, z_1]$  with  $\|z_2 - x\| = \rho_i - \eta/2$ . There is a  $z_3 \in D$  such that  $\|z_2 - z_3\| < \eta/2$  and  $\varphi(z_3) \leq m_{i-1}$ . Now, since  $x$ ,  $z_1$ , and  $z_2$  are collinear,

$$\begin{aligned} \|z_1 - z_3\| &\leq \|(z_1 - x) - (z_2 - x)\| + \|z_2 - z_3\| \\ &= \|z_1 - x\| - \|z_2 - x\| + \|z_2 - z_3\| \\ &< (\rho_i + \varepsilon_1 - \eta) - (\rho_i - \eta/2) + \eta/2 = \varepsilon_1. \end{aligned}$$

But  $\varphi(z_1) = \varphi(z_3) + 2\varepsilon_1$ , by definition of  $m_i$ , implying  $f(z_1) - f(z_3) \geq \varepsilon_1$ , and this violates the fact that  $f$  is 1-Lipschitz.  $\square$

Returning to the proof of the lemma, we consider three cases. Recall that  $i = i(K, n)$  and  $\rho_i = \rho_i(K, n)$  unless otherwise stated.

- (1) If for some  $j = j(K, n)$ ,  $\rho_j(K, n) \in [\delta, \varepsilon_1]$ , then define a decreasing, Lipschitz function  $\zeta_n^K = \zeta_1 \in C^\infty(\mathbb{R}, [0, 1])$  such that  $\zeta_1(t) = 1$  for  $t \in [0, \rho_j - \delta/2]$ ,  $\zeta_1(t) = 0$  for  $t \geq \rho_j - \delta/4$ . Pick  $x_{n'}^{K'}$  with  $\|x_n^K - x_{n'}^{K'}\| \in [0, \delta/4]$  and  $\varphi(x_{n'}^{K'}) = m_{j-1}$ . Note that we may assume  $\text{Lip}(\zeta_1) \leq 5/\delta$ .
- (2) If for some  $j$ ,  $\rho_j \in (0, \delta)$ , then define a decreasing, Lipschitz function  $\zeta_n^K = \zeta_2 \in C^\infty(\mathbb{R}, [0, m_{j-1}/m_j])$  such that  $\zeta_2(t) = m_{j-1}/m_j$  for  $t \in [0, \delta + \delta/4]$ ,  $\zeta_2(t) = 0$  for  $t \geq \delta + \delta/2$ . Observe that since  $\rho_j \neq 0$ ,  $j > 1$  and so  $m_{j-1}/m_j$  is well defined. Pick  $x_{n'}^{K'}$  with  $\|x_n^K - x_{n'}^{K'}\| \in [\rho_j, \delta]$  and  $\varphi(x_{n'}^{K'}) = m_j$ . Note that we may assume  $\text{Lip}(\zeta_2) \leq \frac{m_{j-1}}{m_j} \frac{5}{\delta} \leq \frac{m_1}{m_2} \frac{5}{\delta} \leq \frac{5}{\delta}$ .
- (3) If there is no  $j$  such that  $0 < \rho_j \leq \varepsilon_1$ . If  $\rho_j = 0$  with  $j$  the maximal such index and  $j > 1$ , then take  $x_{n'}^{K'} = x_n^K$  and  $\zeta_n^K = \zeta_2$ . If, on the other hand,  $\rho_1 = 0$  is the largest such index, take  $x_{n'}^{K'} = x_n^K$  and  $\zeta_n^K$  of the form  $\zeta_1$  using  $\rho_2$  in its definition. We handle  $\rho_j > \varepsilon_1$  similarly.

We verify that, with the condition on  $y$ , we must have

$$\begin{aligned} & \varphi(x_{n'}^{K'}) \zeta_\delta(\|y - x_n^K\|) \zeta_n^K(\|y - x_{n'}^{K'}\|) \\ & \geq \varphi(x_{m'}^{L'}) \zeta_\delta(\|y - x_m^L\|) \zeta_m^L(\|y - x_{m'}^{L'}\|) \end{aligned}$$

for any  $(L, m) \in (\mathcal{F}, \mathbb{N})$ .

The argument depends on the case:

- (1) If case 1 holds for  $x_n^K$ , then  $\rho_j(K, n) > 0$ ,  $\zeta_\delta(\|y - x_n^K\|) = 1$  and  $\zeta_n^K(\|y - x_{n'}^{K'}\|) = 1$  since  $\|y - x_{n'}^{K'}\| \leq \delta/32 + \delta/4 < \rho_j - \delta/2$ . Suppose that for some  $x_m^L$ ,

$$\begin{aligned} & \varphi(x_{m'}^{L'}) \zeta_\delta(\|y - x_m^L\|) \zeta_m^L(\|y - x_{m'}^{L'}\|) \\ (2.1) \quad & > \varphi(x_{n'}^{K'}) \zeta_\delta(\|y - x_n^K\|) \zeta_n^K(\|y - x_{m'}^{L'}\|) = m_{j-1}. \end{aligned}$$

Then  $\|y - x_m^L\| < \delta/16$ , else  $\zeta_\delta(\|y - x_m^L\|) = 0$ . In particular this shows that  $\|x_n^K - x_m^L\| < \delta/8$ . Now  $\varphi(x_{m'}^{L'}) \geq m_{j+1}$  is untenable. Indeed, suppose this were the case. Since by the Fact above,  $\rho_{j+1}$  cannot be in  $[0, \varepsilon_1]$  it must be that  $\|y - x_{m'}^{L'}\| \geq \varepsilon_1 - \frac{\delta}{32}$ . But then  $\zeta_m^L(\|y - x_{m'}^{L'}\|) = 0$  (regardless of the way that  $\zeta_m^L$  is defined). So if (2.1) holds it must be that  $\varphi(x_{m'}^{L'}) = m_j$ . We consider the subcases.

- If case 1 holds for  $x_m^L$ , then  $\rho_{j^*}(L, m) \in [\delta, \varepsilon_1]$ , where  $j^* = j^*(L, m)$ , and  $\varphi(x_{m'}^{L'}) = m_{j^*-1}$ . As  $\varphi(x_{m'}^{L'}) = m_j$ , we must then have  $j^* = j + 1$ . Let us show that this leads to a contradiction. If  $j^* = j + 1$ , then by the Fact,  $\rho_{j^*}(K, n) \geq \rho_j(K, n) + \varepsilon_1 \geq \delta + \varepsilon_1$ . But  $\|x_n^K - x_m^L\| < \delta/8$  easily implies that  $|\rho_{j^*}(L, m) - \rho_{j^*}(K, n)| < \delta/8$ , and so  $\rho_{j^*}(L, m) > \frac{7\delta}{8} + \varepsilon_1 \notin [\delta, \varepsilon_1]$ , a contradiction.
- If case 2 holds for  $x_m^L$ , then  $m_j = \varphi(x_{m'}^{L'}) = m_{j^*} \Rightarrow j^* = j$ , and since  $\zeta_m^L(t) \leq m_{j^*-1}/m_{j^*} = m_{j-1}/m_j$  for all  $t$ , we must have  $\varphi(x_{m'}^{L'}) \geq m_{j+1}$  which we have observed is untenable.
- If case 3 holds for  $x_m^L$  with  $\rho_{j^*} = 0$  and  $j^* > 1$ , then  $m_j = \varphi(x_{m'}^{L'}) = m_{j^*} \Rightarrow j = j^*$ , and we argue as in case 2 above. If case 3 holds for  $x_m^L$  where  $j^* = 1$ , then  $m_j = \varphi(x_{m'}^{L'}) = m_1$  implying  $j = 1$  and so  $\rho_j(K, m) = 0$ , a contradiction. When  $\rho_{j^*} > \varepsilon_1$ , we proceed similarly.

This completes the case 1 analysis.

(2) If case 2 holds for  $x_n^K$ , then  $\zeta_\delta(\|y - x_n^K\|) = 1$  and  $\zeta_n^K(\|y - x_{n'}^{K'}\|) = m_{j-1}/m_j$  since  $\|y - x_{n'}^{K'}\| \leq \delta/32 + \delta < 5\delta/4$ . Thus

$$\varphi(x_{n'}^{K'}) \zeta_\delta(\|y - x_n^K\|) \zeta_n^K(\|y - x_{n'}^{K'}\|) = m_{j-1}.$$

Suppose that for some  $x_m^L$

$$\varphi(x_{m'}^{L'}) \zeta_\delta(\|y - x_m^L\|) \zeta_m^L(\|y - x_{m'}^{L'}\|) > m_{j-1}.$$

Then we must have  $\varphi(x_{m'}^{L'}) = m_j$  where again  $\varphi(x_{m'}^{L'}) \geq m_{j+1}$  is untenable. From this point the argument is as in case 1.

(3) If  $\rho_j = 0$  with  $j > 1$ , we argue as above for case 2. If  $\rho_j = 0$  with  $j = 1$ , we argue as above for case 1. The case  $\rho_j > \varepsilon_1$  is similar.  $\square$

**Remark 1.** It follows from the proof of Lemma 1, that for any  $x_n^K$  with associated point  $x_{n'}^{K'}$ , and  $y \in X$  with  $\|y - x_n^K\| < \delta/16$ , we have  $\varphi(x_{n'}^{K'}) \zeta_n^K(\|y - x_{n'}^{K'}\|) = m_{j-1}$ , for some  $j = j(K, n)$ .

For the remainder of this note, if  $x_n^K$  is given, we shall denote the associated point in  $\mathcal{D}$  as provided in Lemma 1 by  $x_{n'}^{K'}$ . We also use the quantities  $\varepsilon_1$  and  $\delta$  as defined above, in the sequel. It is worth noting that  $\zeta_\delta(t) \leq 1$  and  $\zeta_n^K \leq 1$  for all  $(K, n)$ ; facts we shall use later. Let us put

$$L = \max \{ \mathbf{Lip}(\zeta_\delta \zeta_1), \mathbf{Lip}(\zeta_\delta \zeta_2) \}.$$

Now we define a coordinatewise  $C^p$  smooth map  $S : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2)$  by,

$$(Sx)_{(K,n,m)} = \varphi \left( x_{n'}^{K'} \right) \zeta_\delta \left( \|x - x_n^K\| \right) \zeta_n^K \left( \|x - x_{n'}^{K'}\| \right)$$

noting that  $(Sx)_{(K,n,m)} \leq \max \varphi \leq 1$ , since  $\zeta_\delta \leq 1$ ,  $\zeta_n^K \leq 1$ .

Let  $\nu \in C^\infty(\mathbb{R}, [0, 1])$  be such that  $\nu(t) = 0$  for  $t \leq 1$ ,  $\nu(t) > 0$  for  $t > 1$ , and  $0 \leq \nu' \leq 3$ .

Now define  $T : X \rightarrow c_0(\mathcal{F} \times \mathbb{N}^2)$  by,

$$(Tx)_{(K,n,m)} = \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) (Sx)_{(K,n,m)}.$$

Let us first see that  $T$  maps into  $c_0(\mathcal{F} \times \mathbb{N}^2)$ . Let  $\varepsilon' > 0$ , fix  $x \in X$  and fix  $N$  with  $1/N < \varepsilon'$ . Now  $\max\{n, m\} > N$  implies  $(Tx)_{(K,n,m)} < \varepsilon'$  since  $\nu \leq 1$  and  $(Sx)_{(K,n,m)} \leq 1$ . Next, by property (iii) of a SPRI, for each  $l \in \mathbb{N}$ , let  $F_l$  be a finite subset of  $\Gamma$  such that  $\alpha \notin F_l$  implies  $\|R_\alpha(x)\| < 1/l$ , which implies  $\nu(l \|R_\alpha(x)\|) = 0$ . Finally, let

$$\mathcal{S} = \{(K, n, m) \in \mathcal{F} \times \mathbb{N}^2 : n, m \leq N, K \subset F_m\}.$$

Then  $\mathcal{S}$  is finite, and  $(K, n, m) \notin \mathcal{S}$  implies  $(Tx)_{(K,n,m)} < \varepsilon'$ .

Next, fix  $(K, n, m)$  and consider the coordinate function  $x \rightarrow (Sx)_{(K,n,m)}$ . Observe that, since  $\|\|\cdot\|\| \leq 1$  and  $\varphi \leq 1$ , we have

$$\begin{aligned} & \left\| (Sx)'_{(K,n,m)} \right\| \\ &= \left\| \varphi \left( x_{n'}^{K'} \right) \left( \zeta_\delta \left( \|x - x_n^K\| \right) \zeta_n^K \left( \|x - x_{n'}^{K'}\| \right) \right)' \right\| \\ &\leq L. \end{aligned}$$

It follows that for any  $x, x' \in X$ ,

$$\left| (Sx)_{(K,n,m)} - (Sx')_{(K,n,m)} \right| \leq L \|x - x'\|,$$

and so  $S : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2)$  is continuous.

Moreover, since each coordinate function  $x \rightarrow (Sx)_{(K,n,m)}$  is Lipschitz with constant independent of  $(K, n, m)$ ,  $S : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2)$  is Lipschitz.



Next we have,

$$\begin{aligned}
(Tx)'_{(K,n,m)} &= \frac{1}{nm|K|} \sum_{\beta \in K} \left[ \prod_{\alpha \in K \setminus \{\beta\}} \nu(m \|R_\alpha(x)\|) \right. \\
&\quad \times \left. \nu'(m \|R_\beta(x)\|) m \|R_\beta(x)\|' (R'_\beta(x)) \right] (Sx)_{(K,n,m)} \\
&\quad + \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) (Sx)'_{(K,n,m)},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\left\| (Tx)'_{(K,n,m)} \right\| \\
&\leq \frac{1}{nm|K|} \sum_{\beta \in K} \left[ \prod_{\alpha \neq \beta \in K} \nu(m \|R_\alpha(x)\|) \right. \\
&\quad \times \left. \nu'(m \|R_\beta(x)\|) m \|R_\beta(x)\|' \|R'_\beta(x)\| \right] |(Sx)_{(K,n,m)}| \\
&\quad + \frac{1}{nm|K|} \left( \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) \|(Sx)'_{(K,n,m)}\| \\
&\leq \frac{1}{nm|K|} \sum_{\beta \in K} 3m \|R_\beta\| + L < 3L + L = 4L,
\end{aligned}$$

where we have used;  $\nu \leq 1$ ,  $\nu' \leq 3$ ,  $\|\cdot\|' \leq 1$ , and  $\|R_\beta\| \leq 1$  for all  $\beta \in \Gamma$ .

Hence for each coordinate,

$$\left| (Tx)_{(K,n,m)} - (Tx')_{(K,n,m)} \right| \leq 4L \|x - x'\|,$$

implying  $T : X \rightarrow c_0(\mathcal{F} \times \mathbb{N}^2)$  is continuous and, as above, we have that each coordinate function  $x \rightarrow (Tx)_{(K,n,m)}$  is Lipschitz with constant independent of  $(K, n, m)$ , and so  $T : X \rightarrow c_0(\mathcal{F} \times \mathbb{N}^2)$  is also Lipschitz.

Now for each fixed  $x \in X$ , by property (iv) of an SPRI, there exists  $x_n^K \in \mathcal{D}$  with  $\|x - x_n^K\| < \delta/32$  and  $R_\alpha(x) \neq 0$  for all  $\alpha \in K$ . It follows from Lemma 1 that for this  $K$  and  $n$ , and any  $m$ ,

$$(Sx)_{(K,n,m)} = \varphi(x_n^{K'}) \zeta_\delta(\|x - x_n^K\|) \zeta_n^K(\|x - x_n^{K'}\|) = \|Sx\|_\infty.$$

Moreover, from the definition of  $\nu$ , for sufficiently large  $m \in \mathbb{N}$  we have  $\nu(m \|R_\alpha(x)\|) > 0$  for all  $\alpha \in K$ , and so for this choice of  $(K, n, m)$ ,

$$\begin{aligned} (Tx)_{(K,n,m)} &= \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) (Sx)_{(K,n,m)} \\ &= \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) \|Sx\|_\infty > 0. \end{aligned}$$

From the observations immediately above, with the same choice of  $(K, n, m)$  for the given  $x$ , it follows that,

$$\begin{aligned} (2.2) \quad \|Sx\|_\infty &= (Sx)_{(K,n,m)} < (Sx)_{(K,n,m)} + \frac{1}{2} (Tx)_{(K,n,m)} \\ &\leq \left\| Sx + \frac{1}{2} Tx \right\|_\infty. \end{aligned}$$

Next, since for any  $x \in X$  there exists  $(K, n, m)$  with  $(Tx)_{(K,n,m)} > 0$ , we have for such  $(K, n, m)$ ,

$$\begin{aligned} (Tx)_{(K,n,m)} &= \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) (Sx)_{(K,n,m)} \leq (Sx)_{(K,n,m)} \\ &< (Sx)_{(K,n,m)} + \frac{1}{2} (Tx)_{(K,n,m)}, \end{aligned}$$

and hence  $(Tx)_{(K,n,m)} < \|Sx + \frac{1}{2} Tx\|_\infty$ . That we may replace  $(Tx)_{(K,n,m)}$  in this last inequality with  $\|Tx\|_\infty$  while maintaining the strictness of the inequality follows from the fact that  $T$  maps into  $c_0(\mathcal{F} \times \mathbb{N}^2)$ ; and therefore we have

$$(2.3) \quad \|Tx\|_\infty < \left\| Sx + \frac{1}{2} Tx \right\|_\infty.$$

We define  $\Phi_F : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2) \oplus c_0(\mathcal{F} \times \mathbb{N}^2)$  by,  $\Phi_F(x) = (Sx, Tx)$ . The inequalities (2.2) and (2.3) show that  $\Phi_F$  maps into  $U = U(\mathcal{F} \times \mathbb{N}^2)$  (see Theorem 1), and moreover, given that both  $T$  and  $S$  are Lipschitz, we have that  $\Phi_F$  is Lipschitz as well.

Next let  $\|(\cdot, \cdot)\|$  be the  $C^\infty$  smooth norm on  $U \subset l_\infty(\mathcal{F} \times \mathbb{N}^2) \oplus c_0(\mathcal{F} \times \mathbb{N}^2)$  as given by Theorem 1. Since as shown  $\Phi_F$  is continuous and maps into

the open subset  $U$ , we have that the composition  $\|\Phi_F(x)\|$  is  $C^p$  smooth given that both  $S$  and  $T$  are coordinatewise  $C^p$  smooth, and on  $U$  the norm  $\|(\cdot, \cdot)\|$  depends locally on only finitely many non-zero coordinates. We note,  $\|\Phi_F(x)\| \leq A \|\Phi_F(x)\|_\infty = A \max\{\|Sx\|_\infty, \|Tx\|_\infty\} \leq A$ , since  $\|Tx\|_\infty \leq \|Sx\|_\infty \leq 1$ .

Now define  $\widehat{S} : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2)$  by

$$(\widehat{S}x)_{(K,n,m)} = \zeta_\delta(\|x - x_n^K\|).$$

Likewise we define

$$(\widehat{T}x)_{(K,n,m)} = \left( \frac{1}{nm|K|} \prod_{\alpha \in K} \nu(m \|R_\alpha(x)\|) \right) (\widehat{S}x)_{(K,n,m)}.$$

We now analogously define  $\Phi : X \rightarrow l_\infty(\mathcal{F} \times \mathbb{N}^2) \oplus_{c_0}(\mathcal{F} \times \mathbb{N}^2)$  by,  $\Phi(x) = (\widehat{S}x, \widehat{T}x)$ . It is easy to see that again  $\Phi$  maps into  $U(\mathcal{F} \times \mathbb{N}^2)$  using an argument similar to that used above for  $\Phi_F$ . Indeed, for  $x$  and  $x_n^K$  with  $\|x - x_n^K\| < \delta/32$ , we have  $\|\widehat{S}x\|_\infty = \zeta_\delta(\|x - x_n^K\|) = 1$ .

Finally define,

$$K(x) = \frac{\|\Phi_F(x)\|}{\|\Phi(x)\|}.$$

As noted above, the numerator of  $K$  is  $C^p$  smooth. Also, for any  $x \in X$  we have  $\|\Phi(x)\| \geq \|(\widehat{S}x, \widehat{T}x)\|_\infty \geq \|\widehat{S}x\|_\infty = 1$ , and so  $K$  is  $C^p$  smooth.

Now, given that  $\Phi_F$  is Lipschitz, so is the composition  $\|\Phi_F(x)\|$ , and similarly for  $\|\Phi(x)\|$ . Since in addition  $\|\Phi_F(x)\|$  is bounded, and the denominator  $\|\Phi(x)\|$  is bounded below by 1, it follows that the quotient function  $K$  is Lipschitz.

We finally show that  $|K - \varphi| < \varepsilon$ . To this end fix  $x \in X$  and let,

$$\mathcal{C} = \{(K, n) \in \mathcal{F} \times \mathbb{N} : \|x - x_n^K\| < \delta/16 < \varepsilon_1 < \varepsilon/3A\}.$$

Note that if  $(K, n) \notin \mathcal{C}$ , then  $(Sx)_{(K,n,m)} = 0$  for all  $m$ , from which it follows that  $\Phi_F(x)_{(K,n,m)} = \Phi(x)_{(K,n,m)} = 0$ .

Now we estimate (using  $\varphi \geq 0$ ),

$$\begin{aligned}
& |K(x) - \varphi(x)| \\
&= \left| \frac{\|\Phi_F(x)\|}{\|\Phi(x)\|} - \varphi(x) \frac{\|\Phi(x)\|}{\|\Phi(x)\|} \right| \\
&= \left| \frac{\|\Phi_F(x)\|}{\|\Phi(x)\|} - \frac{\|\varphi(x)\Phi(x)\|}{\|\Phi(x)\|} \right| \\
&\leq \frac{1}{\|\Phi(x)\|} \left\| \begin{pmatrix} \left( \left( \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right) \zeta_\delta(\|x - x_n^K\|), \right. \\ \left. \left( \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right) (\hat{T}x)_{(K,n,m)} \right) \end{pmatrix} \right\|
\end{aligned}$$

Since only those coordinates in  $\mathcal{C}$  survive, we need only consider  $(K, n, m) \in \mathcal{C} \times \mathbb{N}$ . Recall that this implies  $\|x - x_n^K\| < \delta/16$ . It will suffice to consider case 1 and case 2 from Lemm 1; case 3 being similar. Then by Remark 1,  $\varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) = m_{j-1}$ , for some  $j = j(K, n)$  and all pairs  $(x_n^K, x_{n'}^{K'})$  where  $\|x - x_n^K\| < \delta/16$ . It also follows from the definition of  $\rho_j(K, n)$  that  $\varphi(x_n^K) = m_{j-1}$ . Now as  $\|x - x_n^K\| < \delta/16$ , from the 1-Lipschitz property of  $f$  we also have (when defined) that either  $\varphi(x) = m_{j-2}$ ,  $m_{j-1}$ , or  $m_j$ . In any event, for  $p = 0, 1, 2$ , this gives

$$\left| \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right| = |m_{j-1} - m_{j-p}| < \varepsilon_1 < \varepsilon/3A.$$

Hence, for  $(K, n, m) \in \mathcal{C} \times \mathbb{N}$  we have,

$$\begin{aligned}
& \left| \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right| |\zeta_\delta(\|x - x_n^K\|)| \\
& \leq \left| \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right| < \varepsilon/3A.
\end{aligned}$$

Similarly,

$$\left| \left( \varphi(x_{n'}^{K'}) \zeta_n^K(\|x - x_{n'}^{K'}\|) - \varphi(x) \right) (\hat{T}x)_{(K,n,m)} \right| < \varepsilon/A.$$

Now, as  $1 \leq \|\Phi(x)\|$  and  $\|\cdot\| \leq A \|\cdot\|_\infty$ , the estimates above finally gives us  $|K(x) - \varphi(x)| < \varepsilon$ . ■

**Corollary 1** (GTWZ). *Let  $X$  be a WCG Banach space which admits a  $C^p$  smooth norm. Then  $X$  admits  $C^p$  smooth partitions of unity.*

**Proof.** Let  $A \subset X$  be open and bounded, and set  $\rho(x) = \mathbf{dist}(x, X \setminus A)$ . Note that  $\rho$  is uniformly continuous, and since  $A$  is bounded,  $\rho$  is bounded. Now our Theorem 3 can be applied to  $\rho$  to produce  $C^p$  smooth, uniform approximates. Finally, an examination of the proof of Theorem VIII.3.12 [DGZ] shows that the uniform smooth approximation of such  $\rho$  is sufficient to conclude that  $X$  admits  $C^p$  smooth partitions of unity.  $\square$

The next result is based on [AFK, Lemma 1], which was motivated by [HJ1].

**Theorem 4.** *Let  $X$  be a WCG Banach space which admits a  $C^p$  smooth norm. Then we have:*

- (1) *For every convex subset  $Y \subseteq X$ , every uniformly continuous function  $f : Y \rightarrow \mathbb{R}$ , and every  $\varepsilon > 0$ , there exists a Lipschitz,  $C^p$ -smooth function  $K : X \rightarrow \mathbb{R}$  such that  $|f(y) - K(y)| < \varepsilon$  for all  $y \in Y$ .*
- (2) *There exists a constant  $C_0 \geq 1$  such that, for every subset  $Y \subseteq X$ , every  $\eta$ -Lipschitz function  $f : Y \rightarrow \mathbb{R}$ , and every  $\varepsilon > 0$ , there exists a  $C_0\eta$ -Lipschitz,  $C^p$ -smooth function  $K : X \rightarrow \mathbb{R}$  such that  $|f(y) - K(y)| < \varepsilon$  for all  $y \in Y$ .*

**Proof.** For (1), observe that because  $f$  is real-valued and  $Y$  is convex, by [BL, Proposition 2.2.1 (i)]  $f$  can be uniformly approximated by a Lipschitz map, and hence it is enough to establish (2).

The proof of Theorem 3 shows that there is  $C \geq 1$  such that for every 1-Lipschitz function  $g : X \rightarrow [0, 10]$ , there exists a  $C^p$  function  $\varphi : X \rightarrow \mathbb{R}$  such that

- (1)  $|g(x) - \varphi(x)| \leq 1/8$  for all  $x \in X$
- (2)  $\varphi$  is  $C$ -Lipschitz.

Indeed, the proof of Lemma 1 shows that the Lipschitz constants of the  $\zeta_n^K$  depend only on  $\varepsilon$  and the bound of  $F$ , from which it follows that the Lipschitz constant of the smooth approximate  $K$  in Theorem 3 has a likewise dependence.

We first see that this result remains true for functions  $g$  taking values in  $\mathbb{R}$  if we replace  $1/8$  with  $1$  and we allow  $C$  to be slightly larger. Indeed, by considering the function  $h = \theta \circ \varphi$ , where  $\theta$  is a  $C^\infty$  smooth function  $\theta : \mathbb{R} \rightarrow [0, 10]$  such that  $|t - \theta(t)| \leq 1/4$  if  $t \in [0, 10]$ ,  $\theta(t) = 0$  for  $t \leq 1/8$ , and  $\theta(t) = 10$  for  $t \geq 10 - 1/8$ , we get the following result: there exists  $C_0 := C \text{Lip}(\theta)$  such that for every 1-Lipschitz function  $g : X \rightarrow [0, 10]$  there exists a  $C^p$  function  $h : X \rightarrow [0, 10]$  such that

- (1)  $|g(x) - h(x)| \leq 1/2$  for all  $x \in X$
- (2)  $h$  is  $C_0$ -Lipschitz
- (3)  $g(x) = 0 \implies h(x) = 0$ , and  $g(y) = 10 \implies h(y) = 10$ .

Now, for a 1-Lipschitz function  $g : X \rightarrow [0, +\infty)$  we can write  $g(x) = \sum_{n=0}^{\infty} g_n(x)$ , where

$$g_n(x) = \begin{cases} g(x) - 10n & \text{if } 10n \leq g(x) \leq 10(n+1), \\ 0 & \text{if } g(x) \leq 10n, \\ 10 & \text{if } 10(n+1) \leq g(x) \end{cases}$$

and the sum is locally finite. The functions  $g_n$  are clearly 1-Lipschitz and take values on the interval  $[0, 10]$ , so there are  $C^p$  functions  $h_n : X \rightarrow [0, 10]$  such that for all  $n \in \mathbb{N}$  we have that  $h_n$  is  $C_0$ -Lipschitz,  $|g_n - h_n| \leq 1/2$ , and  $h_n$  is 0 or 10 wherever  $g_n$  is 0 or 10. It is easy to check that the function  $h : X \rightarrow [0, +\infty)$  defined by  $h = \sum_{n=0}^{\infty} h_n$  is  $C^p$  smooth,  $C_0$ -Lipschitz, and satisfies  $|g - h| \leq 1$ . This argument shows that there is  $C_0 \geq 1$  such that for any 1-Lipschitz function  $g : X \rightarrow [0, +\infty)$ , there exists a  $C^1$  function  $h : X \rightarrow [0, +\infty)$  such that

- (1)  $|g(x) - h(x)| \leq 1$  for all  $x \in X$
- (2)  $h$  is  $C_0$ -Lipschitz
- (3)  $g(x) = 0 \implies h(x) = 0$ .

Finally, for an arbitrary 1-Lipschitz function  $g : X \rightarrow \mathbb{R}$ , we can write  $g = g^+ - g^-$  and apply this result to find  $C^p$  smooth,  $C_0$ -Lipschitz functions  $h^+, h^- : X \rightarrow [0, +\infty)$  so that  $h := h^+ - h^-$  is  $C_0$ -Lipschitz,  $C^p$  smooth, and satisfies  $|g - h| \leq 1$ .

Now let us prove the Theorem. By replacing  $f$  with the function

$$x \mapsto \inf_{y \in Y} \{f(y) + \eta\|x - y\|\},$$

which is a  $\eta$ -Lipschitz extension of  $f$  to  $X$ , we may assume that  $Y = X$ . Consider the function  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = \frac{1}{\varepsilon} f(\frac{\varepsilon}{\eta}x)$ . It is immediately checked that  $g$  is 1-Lipschitz, so by the result above there exists a  $C^p$  smooth,  $C_0$ -Lipschitz function  $h$  such that  $|g(x) - h(x)| \leq 1$  for all  $x$ , which implies that the function  $K(y) := \varepsilon h(\frac{\eta}{\varepsilon}y)$  is  $C_0\eta$ -Lipschitz and satisfies  $|f(y) - K(y)| \leq \varepsilon$  for all  $y \in X$ .  $\square$

### Additional Remarks

- (1) The result of [GTWZ] supposes the formally weaker hypothesis that  $X$  admit only a  $C^p$  smooth bump function rather than a  $C^p$  smooth norm. However, as noted earlier, for WCG spaces it is unknown if these conditions are equivalent.
- (2) We note that in general it is not possible to establish Theorem 3 from Corollary 1, since while the existence of  $C^p$  smooth partitions of unity implies (in particular) that uniformly continuous functions can be uniformly approximated by  $C^p$  smooth functions, these latter functions are not generally Lipschitz.
- (3) A version of Theorem 3 for separable  $X$  is given in [AFM] (see also [F1]) which does not use the results of Haydon [H1] or Amir and Lindenstrauss [AL].

**Acknowledgment** The authors wish to thank Richard Smith and Luis Sánchez González for reading over the manuscript carefully and making many suggestions which have improved this note.

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